

# Connectedness by Computable Arcs in Effective Peano Continua

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# Motivation and goal

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Classical result of point-set topology (studied for my thesis):

*In Peano continua, any two points are connected by an arc.*

We want to prove a computable version of this result:

*In **effective** Peano continua, any two **computable** points are connected by a **computable** arc.*

# Involving computable topology

Setting: computable metric space  $(X, d, \alpha)$

- has a dense **effectively-separating sequence**  $(\alpha_n)_n$  (its elements: **effective-separating points**) such that “distance between two effectively-separating points” is a recursive function

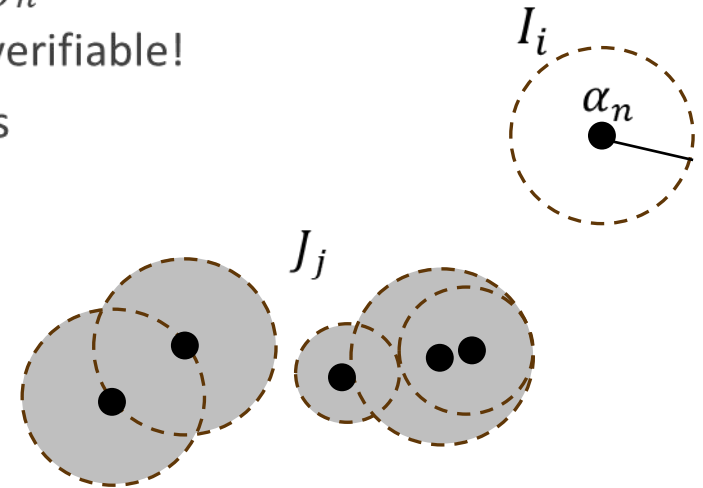
Starting from effectively-separating points to move around the space with, we construct:

- **Rational open balls**  $(I_i)_{i \in \mathbb{N}}$  = all open balls with centres from  $(\alpha_n)_n$  and rational radii
- Their relative position (containedness, disjointness) is effectively verifiable!
- **Rational open sets**  $(J_j)_{j \in \mathbb{N}}$  = all finite unions of rational open balls
- Their relative position and size is effectively verifiable if checking the position of their component rational open balls:  
**formal disjointness, formal containedness, formal diameter**

$$J_{j_1} \diamond_F J_j$$

$$J_{j_1} \subseteq_F J_{j_2}$$

$$\mathbf{fdiam}(J_j) = r$$



# Relevant terms of computability

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**Computable point** = has an effective procedure with which to be approximated by an effectively-separating point at any precision.

**Computable set** = compact set that has an effective procedure with which to be approximated by a finite set of effectively-separating points at any precision.

**Effectively compact space** = computable metric space that is itself a computable set

**Effective continuum** = effectively compact + connected

**Effectively locally connected space** = has a sequence  $(D_j)_j$  of nonempty connected open sets that is computably equivalent to the sequence of rational open balls  $(I_i)_i$ , meaning that elements of these sequences “effectively build each other” as set unions

**Effective Peano continuum** = effective continuum + effectively locally connected

# Full statement of main result

Let  $(X, d, \alpha)$  be an effective Peano continuum and let  $a, b \in X$ ,  $a \neq b$ , be computable points. Then there exists a computable arc between  $a$  and  $b$ .

Outline of proof:

1. Effectively make a sequence of formal chains(1) with connected links(2) from  $a$  to  $b$ (3) which are shrinking(4) and have their links nested “in order”(5).
2. Show that the intersection of all closures of unions of chains from that sequence is
  - a) an arc between  $a$  and  $b$  (lengthy topological argument)
  - b) a computable set (technical details)

# Formal open sets and formal chains

**Formal open sets**  $(\tilde{D}_j)_j$  = finite unions of base formal connected open sets  $(D_j)_j$   $\tilde{D}_{j_1} \diamond_F \tilde{D}_{j_2}$   $\tilde{D}_{j_1} \subseteq_F \tilde{D}_{j_2}$

By proxy of rational open sets and computable equivalence, we can also define **formal disjointness**, **formal containedness**, and **formal boundedness** to check their relative position and size effectively.

**Formal chains**  $(\mathcal{H}_l)_l$  = links are formal open sets from  $(\tilde{D}_j)_j$ , adjacent links intersect, non-adjacent links formally disjoint.

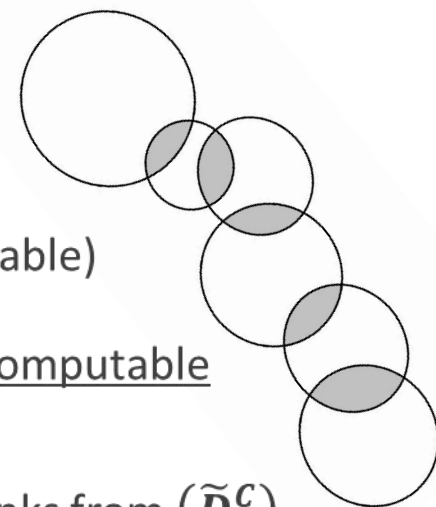
We want chains with connected links: **connected formal open sets**  $(\tilde{D}_j^c)_j$  (effectively enumerable)

It can be shown that having connected links and “being a chain from computable point  $a$  to computable point  $b$ ” are recursively enumerable properties for formal chains:

**Formal connected chains from  $a$  to  $b$**   $(\mathcal{H}_l^{c,a,b})_l$  = formal chains from  $a$  to  $b$  with connected links from  $(\tilde{D}_j^c)_j$ .

**Formal boundedness of link diameter** for formal chains:

$$\tilde{D}_j <_F r$$



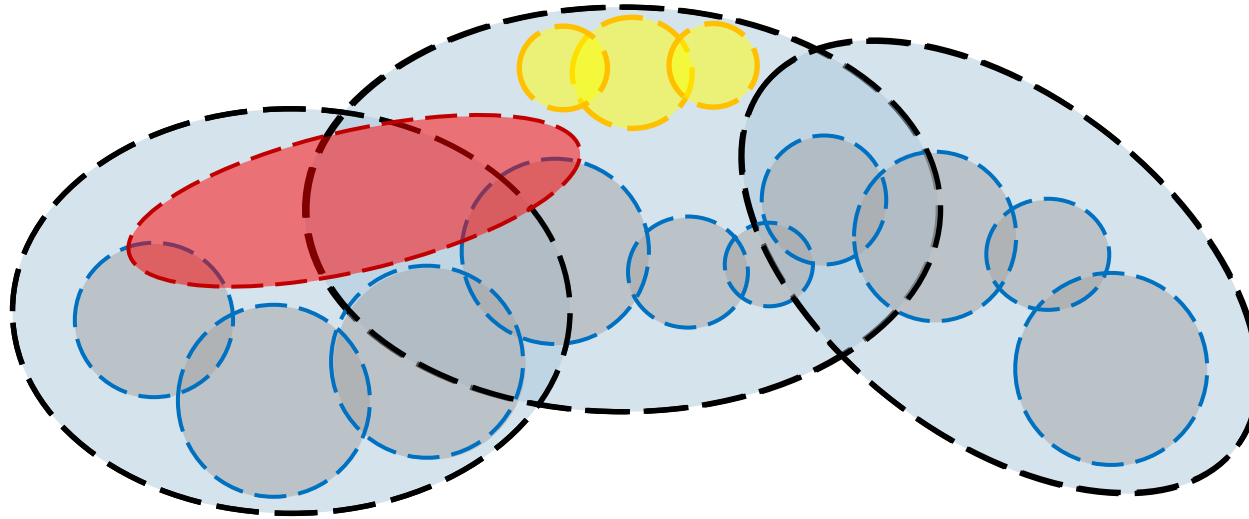
# Chain refinement

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We say of formal chains  $\mathcal{H}_m$  and  $\mathcal{H}_l$  that:

$\mathcal{H}_m$  **refines**  $\mathcal{H}_l$ , denoted  $\mathcal{H}_m \leq \mathcal{H}_l$ , if each link of  $\mathcal{H}_m$  is formally contained in some link of  $\mathcal{H}_l$ ;

**strictly**, denoted  $\mathcal{H}_m < \mathcal{H}_l$ , if the first link and last link of  $\mathcal{H}_m$  are formally contained in the first and last link of  $\mathcal{H}_l$ , respectively.

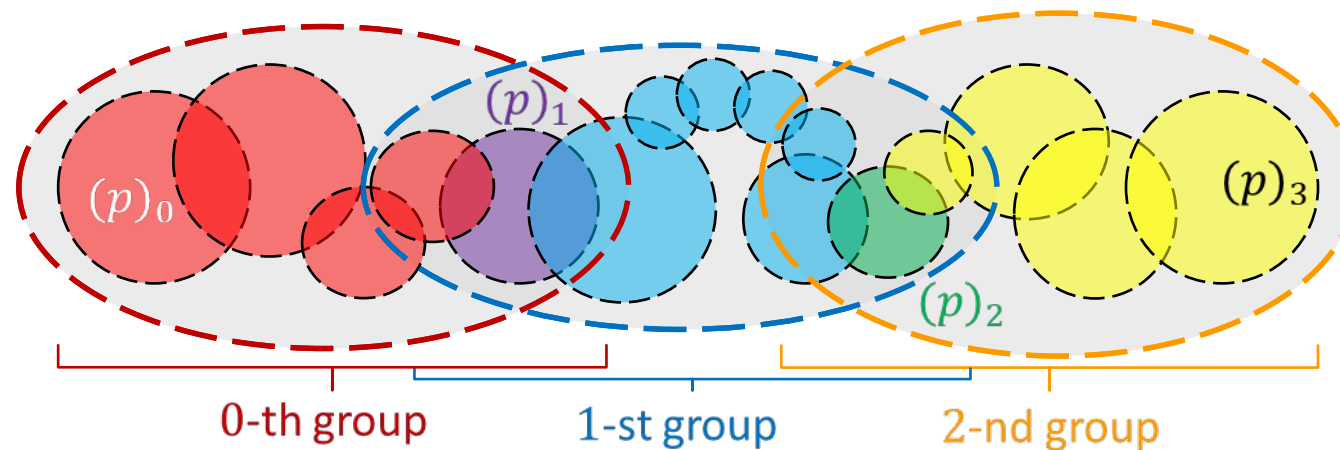


# Fine strict refinement of formal chains

$\mathcal{H}_m$  **finely** strictly refines  $\mathcal{H}_l$ , denoted  $\mathcal{H}_m \ll \mathcal{H}_l$ , if there exists a  $p \in \mathbb{N}$  such that

1.  $0 = (p)_0 < (p)_1 < \dots < (p)_{i+1} = \bar{m}$ 
  - (indices of  $\mathcal{H}_m$ 's links can be ordered into [length of  $\mathcal{H}_l$ ] groups, adjacent groups sharing 1 link...)
2.  $\forall r \in \{0, \dots, \bar{l}\} \ (p)_r \leq i \leq (p)_{r+1} \Rightarrow \tilde{D}_{(m)_i} \subseteq_F \tilde{D}_{(l)_r}$ 
  - (such that the whole  $r$ -th group is formally contained in the  $r$ -th link of chain  $\mathcal{H}_l$ )

It can be shown that this property is effectively enumerable!



This ensures:  
 $\mathcal{H}_m$  goes through the links of  $\mathcal{H}_l$  in order.  
 It does not get out of a link entirely and then go back inside; can't go through the bigger chain "backwards".

# Full statement of main result

Let  $(X, d, \alpha)$  be an effective Peano continuum and let  $a, b \in X$ ,  $a \neq b$ , be computable points. Then there exists a computable arc between  $a$  and  $b$ .

Outline of proof:

1. Effectively make a sequence of formal chains(1) with connected links(2) from  $a$  to  $b$ (3) which are shrinking(4) and have their links nested “in order”(5).
2. Show that the intersection of all closures of unions of chains from that sequence is
  - a) an arc between  $a$  and  $b$  (lengthy topological argument due to 1.)
  - b) a computable set (technical details)

# Result for proof 1:

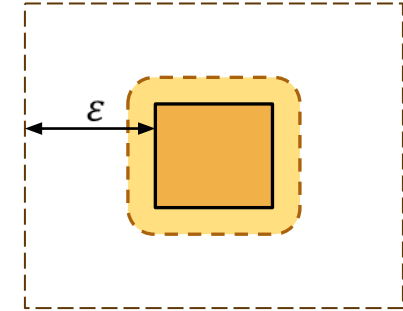
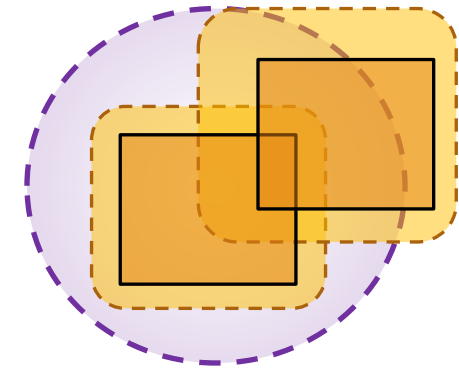
## Connected inflation lemma

**Lemma.** Let  $(X, d, \alpha)$  be an effectively locally connected computable metric space with base connected formal open sets  $(D_j)_j$  and:

- $\mathcal{K}$  a nonempty finite family of connected compact sets,
- $\mathcal{C}$  a finite (possibly empty) set of indices of formal open sets from  $(\tilde{D}_j)_j$ , and
- $\varepsilon > 0$ .

Then every set  $K \in \mathcal{K}$  can be assigned a connected formal open set index  $d_K \in \mathbb{N}$  such that:

1.  $K \subseteq \tilde{D}_{d_K}^{\mathcal{C}}$
2.  $\tilde{D}_{d_K}^{\mathcal{C}} <_F \text{diam}(K) + \varepsilon$
3. If  $e \in \mathcal{C}$  and  $K \subseteq \tilde{D}_e$ , then  $\tilde{D}_{d_K}^{\mathcal{C}} \subseteq_F \tilde{D}_e$
4. If  $K_1$  and  $K_2$  are disjoint, then  $\tilde{D}_{d_{K_1}}^{\mathcal{C}} \diamond_F \tilde{D}_{d_{K_2}}^{\mathcal{C}}$

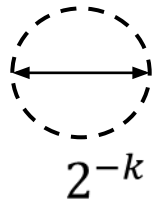
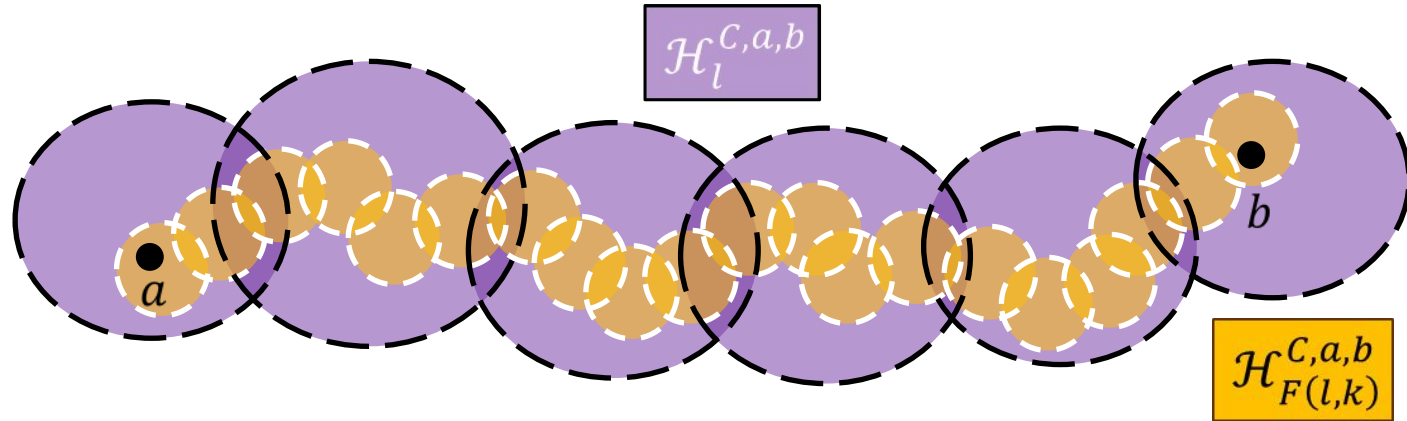


# Result for proof: Finely strictly refining chain lemma

**Lemma.** Let  $(X, d, \alpha)$  be an effective Peano continuum, and let  $a, b \in X$ ,  $a \neq b$ , be computable points.

Then there exists a recursive function  $F: \mathbb{N}^2 \rightarrow \mathbb{N}$  such that, if  $l \in \mathbb{N}$  is any index of a formal connected chain from  $a$  to  $b$  and if  $k \in \mathbb{N}$  is a size-limiting parameter,  $F(l, k)$  is an index of a formal connected chain from  $a$  to  $b$  such that:

1.  $\mathcal{H}_{F(l,k)}^{C,a,b} <_F 2^{-k}$
2.  $\mathcal{H}_{F(l,k)}^{C,a,b} \ll \mathcal{H}_l^{C,a,b}$ .



# Proof sketch: Constructing the first chain

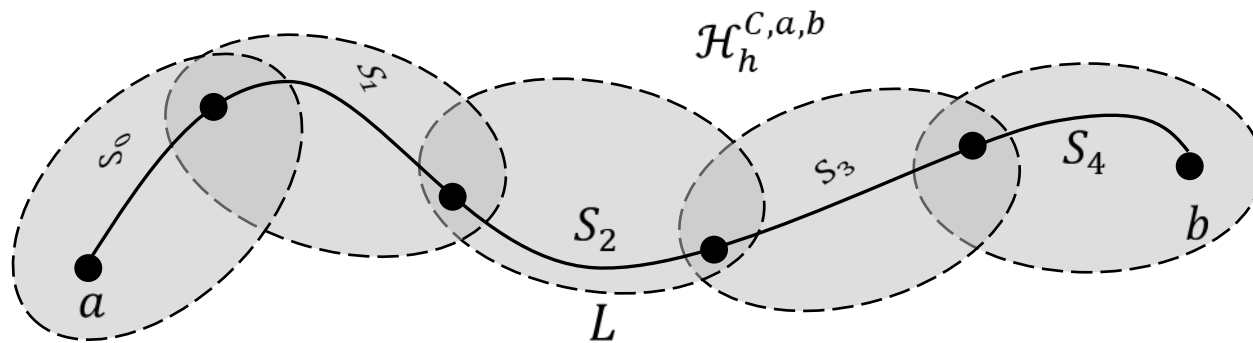
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Effective Peano continuum  $\Rightarrow$  also a regular Peano continuum.

Classical result: there is an arc  $L$  connecting  $a$  and  $b$ !

We subdivide  $L$  into  $N$  subarcs,  $S_k$ ,  $k \in \{0, \dots, N - 1\}$ , such that their diameters are less than  $\frac{1}{2}$ .

We inflate them into a formal connected chain  $\mathcal{H}_h^{C,a,b}$  using the connected inflation lemma:



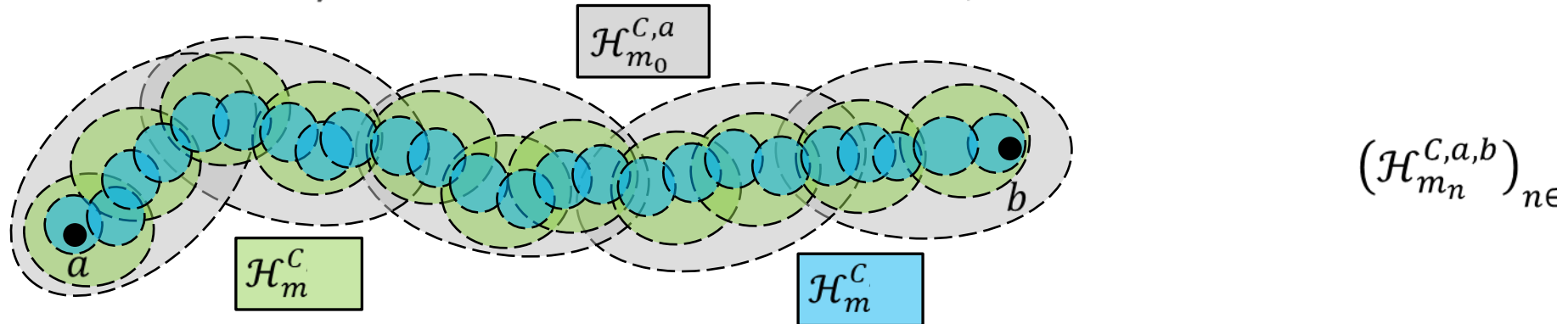
# Sequence of chains

Finely strictly refining chain theorem: recursive finely strictly refining function  $F: \mathbb{N}^2 \rightarrow \mathbb{N}$ .

We define our suitable nested chain sequence function  $m: \mathbb{N} \rightarrow \mathbb{N}$  by recursion as follows:

1.  $m(0) = m_0 = h \iff \mathcal{H}_{m_0}^{C,a,b} = \mathcal{H}_h^{C,a,b}$
2.  $m(n+1) = m_{n+1} = F(m(n), n+1) \iff \mathcal{H}_{m_{n+1}}^{C,a,b} = \mathcal{H}_{F(m(n), n+1)}^{C,a,b}$

As a function defined by recursion with recursive functions, it is itself recursive.



# Candidate for computable arc

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$(\mathcal{H}_{m_n}^{C,a,b})_{n \in \mathbb{N}}$  is a recursive sequence of nested, finely strictly refining chains with link diameters formally bounded by numbers converging to 0.

We define

$$A = \bigcap_{n \in \mathbb{N}} \left( \overline{\bigcup \mathcal{H}_{m_n}^{C,a,b}} \right) = \bigcap_{n \in \mathbb{N}} \left( \overline{\tilde{D}_{(m_n)_0}^C \cup \dots \cup \tilde{D}_{(m_n)_{\overline{m_n}}}^C} \right)$$

(intersection of all closures of unions of chains indexed by sequence  $(m_n)_n$ ).

It can be shown that:

- a)  $A$  is an arc.
- b)  $A$  is a computable set.  $\square$

# $A$ is an arc

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*Proof almost the same as in the classical version of the result, broad strokes:*

It can be shown that  $A$  is a continuum (closed, bounded, connected).

We use the following characterisation of arcs in metric continua:

**Definition.** If  $S$  is a connected topological space, we say that the point  $x \in S$  is a **cut point** of  $S$  if the set  $S \setminus \{x\}$  is disconnected. Otherwise, we say it is a **non-cut point**.

**Theorem.** If  $X$  is a continuum, it is an arc if and only if it has exactly two non-cut points.

It can be shown by several pages of topological arguments that  $a$  and  $b$  are non-cut points in  $A$ , while all other points in  $A$  are cut points, making  $A$  an arc.

# A is a computable set

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*Proof technical, broad strokes:* Starting with  $n \in \mathbb{N}$ , we need to effectively find a finite set of effective separating points that are  $2^{-n}$ -close to  $A$ .

We apply the recursive function  $m(n)$  and use it to effectively retrieve, from effective enumeration of formal connected chains from  $a$  to  $b$ , the chain  $\mathcal{H}_{m_n}^{C,a,b}$  and the numbers  $\{(m_n)_0, \dots, (m_n)_{\overline{m_n}}\}$  that index its links  $\{\tilde{D}_{(m_n)_0}^C, \dots, \tilde{D}_{(m_n)_{\overline{m_n}}}^C\}$ .

Then we effectively find  $\{i_{g((m_n)_0)}, \dots, i_{g((m_n)_{\overline{m_n}})}\}$ , where  $g: \mathbb{N} \rightarrow \mathbb{N}$  is some recursive function due to computable equivalence, such that  $\{I_{g((m_n)_0)}, \dots, I_{g((m_n)_{\overline{m_n}})}: I_{g((m_n)_k)} \subseteq \tilde{D}_{(m_n)_k}^C, \forall k \in \{0, \dots, \overline{m_n}\}\}$ , and then we effectively retrieve the centres of those rational open balls,  $\{\lambda_{g((m_n)_0)}, \dots, \lambda_{g((m_n)_{\overline{m_n}})}\}$ , which are effective separating points contained within each link of  $\mathcal{H}_{m_n}^{C,a,b}$  (which are less than  $2^{-n}$  in diameter) and therefore  $2^{-n}$ -close to  $A \subseteq \mathcal{H}_{m_n}^{C,a,b}$ .  $\square$

# Further plans for looking into:

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Conjecture: *In an effective Peano continuum, any arc can be arbitrarily closely approximated by a computable arc.*

Holds if one can encase a continuum in some  $\varepsilon$ -close effective continuum, needs technical checking

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Thank you!

